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Observations on the moduli space of two dimensional string theory

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Abstract: We explore the moduli space of the two dimensional fermionic string with linear dilaton. In addition to the known 0A and 0B theories, there are two theories with chiral GSO projections, which we call IIA and IIB. They are similar to the IIA and IIB theories of ten dimensions, but are constructed with a different GSO projection. Compactifying these theories on various twisted circles leads to eight lines of theories. Three of them, 0A on a circle, super-affine 0A and super-affine 0B are known. The other five lines of theories are new. At special points on two of them we find the noncritical superstring.

Keywords: Superstring Vacua, Conformal Field Models in String Theory.

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Contents

1. Introduction

There are several reasons to study low dimensional string theories. Since they are simpler than their higher dimensional counterparts, they are often solvable. The exact solution can teach us about new phenomena which could be present also in more generic situations. In particular, these theories involve a linear dilaton and are very similar to more complicated theories with a linear dilaton like the NS5-brane theory. It is important to add that despite their simplicity, these theories are very rich and exhibit many qualitative phenomena which are present in more generic theories.

Here we will discuss the two dimensional theory with worldsheet supersymmetry.[1] The target space is parametrized by two coordinates X and ϕ. The string metric is flat and the dilaton is linear in ϕ. We will limit ourselves to theories which are translation invariant in X. The extension to other theories, including various orbifolds of our theories is straightforward. For most of our discussion we will focus on the weak coupling end of the target space $\phi \to -\infty$. Although various deformations of the background like nonzero NS-NS "tachyon" fields or RR fluxes are possible, they are negligible in that region and the worldsheet theory can be analyzed there using free field methods.

There are four theories with noncompact X. Two of them, the 0A and the 0B theories, are familiar (see, e.g. [4, 5] and references therein). The other two theories, which are less known, can be called IIA and IIB.[2] The two type-II theories are similar to their

[1] For early reviews of the bosonic version of these theories see, e.g. [1, 2, 3].

[2] Motivated by the noncritical superstring construction of [6] (see also the review [7]), the spectrum of these theories in noncompact space was guessed in [8] and mentioned in [9, 10, 11, 12]. Our discussion below will derive this spectrum, and will clarify its connection to the noncritical superstring.

ten dimensional analogs. The spectrum of the IIA theory is not chiral. It consists of a single Majorana fermion with its left and right moving components. The IIB theory is chiral. Its spectrum has two Majorana Weyl fermions of one chirality and a scalar of the opposite chirality, thus cancelling the anomalies in a nontrivial way. However, unlike their ten dimensional counterparts, these theories do not have spacetime supersymmetry. More microscopically, it is important to stress that the GSO projection in these two dimensional theories is different than in ten dimensions. In section 2 we describe the worldsheet construction of these theories and discuss the relations between them. We also present their ground rings and the puzzles the new theories pose. Perhaps these puzzles, which are reminiscent of black hole physics, could be resolved by assuming that the semiclassical picture is not precise, and that there are more asymptotic states in addition to the perturbative quanta.

In section 3 we examine various twisted circle compactifications of the four noncompact theories. This moduli space is similar to its ten dimensional relative (see figure 1). It includes eight lines parametrized by the compactification radius R. Three of these lines (lines 1 – 3 in figure 1) are known from the study of the type 0 theory. The other five lines (lines 4 – 8 in figure 1) are new. Two special points (marked with black squares in figure 1), one on line 7 and the other on line 8 correspond to the noncritical superstring of [6]. Four other special points (denoted by black circles in figure 1) have enhanced nonabelian symmetry.

1.1 Notations and symmetries

As preparation for our discussion, let us present our notations and symmetries.

As a free worldsheet field, X can be written as a sum of worldsheet chiral components $X = x+\overline{x}$. The fermionic partners of x and $\overline{x}$ are ψ_x and $\overline{\psi}_x$. Together with the superpartners of ϕ they can be bosonized to worldsheet chiral scalars H and $\overline{H}$. Throughout this paper we set $\alpha' = 2$. If needed, dimensions can be restored by multiplying every parameter with dimensions of length by $\sqrt{\alpha'/2}$.

We use four discrete transformations which act on the various fields as follows:

$$\begin{array}{lll} (-1)^{f_L} & \varphi \to \varphi + i\pi\,; & H \to H + \pi \\ (-1)^{F_L} & \varphi \to \varphi + 2\pi i & \\ (-1)^{f_R} & \overline{\varphi} \to \overline{\varphi} + i\pi\,; & \overline{H} \to \overline{H} + \pi \\ (-1)^{F_R} & \overline{\varphi} \to \overline{\varphi} + 2\pi i & \end{array} \tag{1.1}$$

where φ and $\overline{\varphi}$ are the left and right moving bosonized superghosts. $f_{L,R}$ are the left and right moving worldsheet fermion numbers and $F_{L,R}$ are the left and right moving spacetime fermion numbers. Since all our operators will be constructed out of building blocks of the form $e^{-\varphi+inH}$ and $e^{-\frac{\varphi}{2}+i(n+\frac{1}{2})H}$ with $n \in \mathbb{Z}$ (and similarly for the right movers), all the operators in (1.1) square to one when they act on our vertex operators. Therefore, (1.1) are $\mathbb{Z}_2$ transformations.

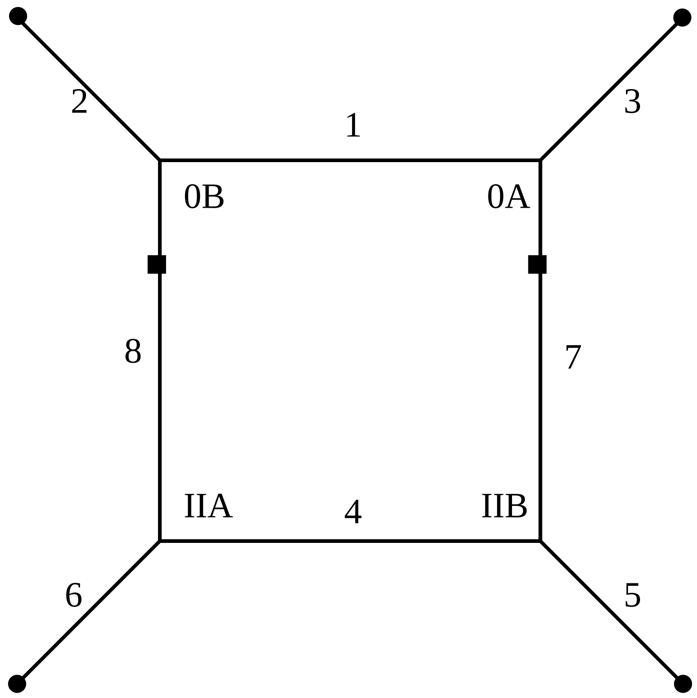

Figure 1: The moduli space of theories we consider. The four corners of the square represent the four theories before compactification. They are labelled by 0B, 0A, IIB and IIA. The eight lines, labelled by $1-8$ represent different compactifications. They are labelled by their number in the text. The points on each line represent compactifications with different radii R. Lines 1, 4, 7 and 8 interpolate between different noncompact theories as R varies between 0 and ∞. Each point on lines 2, 3, 5 and 6 corresponds to two different dual radii; the four points labelled by black circles at the ends of these lines are the selfdual points. At these points the theory has enhanced nonabelian symmetry. The points marked with black squares on lines 7 and 8 represent the noncritical superstrings of [6].

In addition to (1.1) we will also be interested in the two "parity transformations"

$$\begin{aligned} P_{ws}: &\quad z \leftrightarrow \bar{z}\,; &\quad \varphi \leftrightarrow \bar{\varphi}\,; &\quad H \leftrightarrow \overline{H}\,; &\quad x \leftrightarrow \bar{x} \\ P_{ts}: &\quad H \to -H\,; &\quad \overline{H} \to -\overline{H}\,; &\quad x \to -x\,; &\quad \bar{x} \to -\bar{x} \end{aligned} \tag{1.2}$$

P_{ws} is worldsheet parity and P_{ts} is the target space parity. Conjugation by them acts on (1.1). Clearly,

$$\begin{aligned} (-1)^{F_{L,R}} &= P_{ws}(-1)^{F_{R,L}}P_{ws} \\ (-1)^{f_{L,R}} &= P_{ws}(-1)^{f_{R,L}}P_{ws} \end{aligned} \tag{1.3}$$

and using the fact that (1.1) are $\mathbb{Z}_2$ transformations, we find

$$\begin{aligned} (-1)^{F_{L,R}} &= P_{ts}(-1)^{F_{L,R}}P_{ts} \\ (-1)^{F_{L,R}+f_{L,R}} &= P_{ts}(-1)^{f_{L,R}}P_{ts}\,. \end{aligned} \tag{1.4}$$

2. Theories on $\mathbb{R}^2$

There are four theories in noncompact space. Each of them is characterized by a $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of the four $\mathbb{Z}_2$ in (1.1), which acts trivially on all the operators. The remaining $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts as symmetry. More precisely, the group G is a quotient of the four $\mathbb{Z}_2$ in (1.1) by H; i.e. the elements of G can be replaced by elements of G times elements of H.

The four theories are related by orbifolding. Orbifolding each theory by various subgroups of its symmetry G leads to other theories. There are two well known subtleties in doing that. First, in addition to the $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry the theories also have symmetries generated by the parity transformations P_{ws} and P_{ts} which act on G as in (1.3) (1.4). Therefore, conjugation by P_{ws} and P_{ts} restricts the number of distinct orbifolds by subgroups of G. Second, one has to be careful about the sign of the orbifold projection in the twisted sectors. Normally, it is chosen by consistency. Here, since we have an explicit realization in terms of free fields, we simply have to impose the correct generators to mod out by. More explicitly, we can always combine the orbifolding generator with an element of the group $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ which acts trivially. Then we simply mod out by the elements which act trivially in the resulting theory.

2.1 0B

Here the transformations $(-1)^{F_L+F_R} = (-1)^{f_L+f_R} = 1$ act trivially, and the symmetry G is generated by $(-1)^{F_L}$ and $(-1)^{f_L}$. The physical operators, ignoring for the moment operators at special momenta are

$$\begin{aligned}
T(p) &= e^{-\varphi-\overline{\varphi}+ip(x+\overline{x})+(1-|p|)\phi} \\
C(p) &= e^{-\frac{\varphi}{2}-\frac{\overline{\varphi}}{2}+\frac{i\epsilon(p)}{2}(H+\overline{H})+ip(x+\overline{x})+(1-|p|)\phi} \qquad p \neq 0 \\
C_\pm &= e^{-\frac{\varphi}{2}-\frac{\overline{\varphi}}{2}\pm\frac{i}{2}(H+\overline{H})+\phi} \\
\epsilon(p) &= \mathrm{Sign}(p)\,.
\end{aligned} \tag{2.1}$$

Here and below, the absolute value of p in the ϕ dependence follows from the bound on the Liouville exponent [13]. Linear combinations of $C_\pm$ are the zero modes of C and its dual. The propagating particles are the NS-NS "tachyon" T and the R-R scalar C. The symmetry $(-1)^{f_L}$ exchanges C and its dual, and therefore C must be compact and its radius is the selfdual radius [4]. The theory is also invariant under the worldsheet and target space parities P_{ws}, P_{ts}.

In addition to the vertex operators (2.1), the theory also has special operators which exist only at certain discrete momenta. For example,

$$\begin{aligned}
j^0 &= e^{-\varphi}\psi_x \\
\overline{j}^0 &= e^{-\overline{\varphi}}\overline{\psi}_x
\end{aligned} \tag{2.2}$$

are worldsheet currents associated with a U(1) $\times$ U(1) symmetry. Their sum is the momentum p in (2.1) and their difference does not act on (2.1) (it is the winding symmetry of the compactified theory). Using these operators we can construct the two dimensional version of the higher dimensional dilaton/graviton

$$V = j^0 \bar{j}^0 \,. \tag{2.3}$$

Other operators of interest are the ground ring operators [14]–[21]. These are dimension $(0,0)$ vertex operators. The ring is freely generated by

$$R_\pm = e^{-\frac{\varphi}{2} - \frac{\overline{\varphi}}{2} \mp \frac{i}{2}(H+\overline{H}) \pm \frac{i}{2}(x+\overline{x}) - \frac{\phi}{2}} + \cdots \tag{2.4}$$

The vertex operators (2.1) form modules of the ring [20, 4].

$$\begin{aligned} R_+ C(p) &= \begin{cases} T(p+\frac{1}{2}) & p>0 \\ 0 & p<0 \end{cases} \\ R_- C(p) &= \begin{cases} 0 & p>0 \\ T(p-\frac{1}{2}) & p<0 \end{cases} \\ R_+ T(p) &= \begin{cases} -p^2 C(p+\frac{1}{2}) & p>0 \\ 0 & p<0 \end{cases} \\ R_- T(p) &= \begin{cases} 0 & p>0 \\ -p^2 C(p-\frac{1}{2}) & p<0 \end{cases} \end{aligned} \tag{2.5}$$

(the factors of $-p^2$ in the action on $T(p)$ arise from picture changing.) In all these modules we have the relation $R_+R_- = 0$. When a zero momentum tachyon is turned on by adding $\mu T(0)$ to the worldsheet lagrangian, there are still no relations in the ring, but the module relation is deformed to [20, 4]

$$R_+ R_- = \mu \,. \tag{2.6}$$

In the matrix model description of these theories, the fields T and C are interpreted as ripples on the Fermi sea, and the relation (2.6) is interpreted as the shape of the Fermi sea.

Besides turning on the tachyon operator $\mu T(0)$, we can also preserve translation invariance while turning on the RR operators

$$q_+ C_+ + q_- C_- = q(C_+ + C_-) + \widetilde{q}(C_+ - C_-) \,. \tag{2.7}$$

It is not easy to use worldsheet methods to compute the consequences of such deformations, but this should be doable using matrix models.

Finally, let us mention the action of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ on these deformations

$$\begin{aligned} &(-)^{F_L} && q_\pm \to -q_\pm \\ &(-1)^{f_L} && \mu \to -\mu\,; \qquad q_\pm \to \pm q_\pm \,. \end{aligned} \tag{2.8}$$

2.2 0A

An orbifold of the 0B theory by its symmetry $(-1)^{F_L}$ leads to the 0A theory. More precisely, in the 0A theory the transformations $(-1)^{F_L+F_R} = (-1)^{f_L+f_R+F_L} = 1$ act trivially, and $(-1)^{F_L}$ and $(-1)^{f_L}$ generate the symmetry G. The spectrum includes

$$\begin{aligned} T(p) &= e^{-\varphi - \overline{\varphi} + ip(x+\overline{x}) + (1-|p|)\phi} \qquad p \in \mathbb{R} \\ F^\pm &= e^{-\frac{\varphi}{2} - \frac{\overline{\varphi}}{2} \pm \frac{i}{2}(H - \overline{H}) - \phi} \,. \end{aligned} \tag{2.9}$$

Here $F^{\pm}$ are the two RR fluxes of the theory. They are the zero momentum modes of two different gauge fields. The only propagating particle is the "tachyon" T. As in the 0B theory, the theory is also invariant under the two parity transformations P_{ws} and P_{ts} (1.2).

As in the 0B theory we have the currents (2.2) and the dilaton/graviton (2.3). The ground ring, however, is a quotient of the 0B ground ring by $(-1)^{F_L}$; i.e. it includes only the even powers of $R_{\pm}$. It is generated by $O_+ = R_+^2$, $O_- = R_-^2$ and $O_0 = R_+R_-$ with the relation $O_+O_- = O_0^2$. Clearly, the tachyons $T(p)$ are in modules of the ring with the relation $O_0 = 0$.

Again, as in the 0B theory there are three translation invariant deformations

$$\mu T(0) + q_+F^+ + q_-F^- = \mu T(0) + q(F^+ + F^-) + \widetilde{q}(F^+ - F^-) \tag{2.10}$$

and the action of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ on the deformations is as in (2.8). The deformations (2.10) deform the module relation $O_0 = 0$ to $O_0 = \mu$.

2.3 IIB

An orbifold of the 0B theory by its symmetry $(-1)^{f_L}$ leads to the IIB theory. More precisely, here the transformations $(-1)^{f_L+f_R} = (-1)^{f_L+F_L+F_R} = 1$ act trivially, and G is generated by $(-1)^{F_L}$ and $(-1)^{F_R}$. The spectrum includes

$$\begin{aligned}
\Psi_-(p) &= e^{-\frac{\varphi}{2}-\overline{\varphi}-i\frac{H}{2}+ip(x+\overline{x})+(1-|p|)\phi} && p \le 0\\
\overline{\Psi}_-(p) &= e^{-\varphi-\frac{\overline{\varphi}}{2}-i\frac{\overline{H}}{2}+ip(x+\overline{x})+(1-|p|)\phi} && p \le 0\\
C_+(p) &= e^{-\frac{\varphi}{2}-\frac{\overline{\varphi}}{2}+\frac{i}{2}(H+\overline{H})+ip(x+\overline{x})+(1-|p|)\phi} && p \ge 0\,.
\end{aligned} \tag{2.11}$$

The spectrum of particles is a left moving boson C_+ and two right moving Majorana Weyl fermions $\Psi_-(p)$ and $\overline{\Psi}_-(p)$. The chiral nature of the spectrum originates from the absolute value of p in the exponent of ϕ, which arises from the bound of [13]. The theory has only one RR-flux, $C_+(0)$. The theory is invariant under the worldsheet parity transformation P_{ws} but not under P_{ts}. Clearly, $(-1)^{F_{L,R}}$ are related by conjugation by this symmetry. Here, an isomorphic theory (IIB') can be obtained by the action of the target space parity operation P_{ts}.

It is important that the projections in the R-NS and NS-R sectors are opposite to the ten dimensional ones, where the projections are $(-1)^{f_L} = (-1)^{f_R} = 1$. Had we used such a projection, there would not have been any fermions and the spectrum would have included only the left moving boson C_+. This spectrum is anomalous and the theory is likely to be inconsistent.

Unlike the type 0 theory, here there is only one possible translation invariant deformation — the RR flux $q_+C_+(0)$.

As in the 0B theory, we have the operators (2.2) (2.3). The ground ring, however, is a quotient of the 0B ground ring by $(-1)^{f_L}$. It is freely generated by R_- and $O_+ = R_+^2$. The vertex operators (2.11) are again in modules of the ring

$$R_-\Psi_-(p) = ip\overline{\Psi}_-\left(p - \frac{1}{2}\right)$$

$$
\begin{aligned}
R_-\overline{\Psi}_-(p) &= ip\Psi_-\left(p-\frac{1}{2}\right) \\
R_-C_+(p) &= 0 \\
O_+\Psi_-(p) &= 0 \\
O_+\overline{\Psi}_-(p) &= 0 \\
O_+C_+(p) &= -\left(p+\frac{1}{2}\right)^2 C_+(p+1)\,.
\end{aligned} \tag{2.12}
$$

(The factor of ip in the r.h.s. arises from picture changing.) In all these modules we have $R_-O_+ = 0$. By analogy to the known 0A and 0B matrix models, we guess that this theory also has a matrix model, where equation (2.12) means that Ψ_- and $\overline{\Psi}_-$ correspond to ripples on one half of a Fermi surface, while C_+ describes ripples on the other half.

What is the S-matrix of this theory? The incoming particles are the fermions Ψ_- and $\overline{\Psi}_-$, and the outgoing particles are the chiral bosons C_+. Denote the number of Ψ_- and $\overline{\Psi}_-$ incoming quanta by n_- and $\overline{n}_-$ and the number of outgoing C_+ quanta by n_+. Then, the discrete symmetries generated by $(-1)^{F_{L,R}}$ restrict the S-matrix, to obey $n_- + \overline{n}_- \in 2\mathbb{Z}$ and $n_- + n_+ \in 2\mathbb{Z}$. If we turn on background $C_+(0)$, the second restriction is clearly lifted, but the first restriction remains.

This is very peculiar. Consider a process with an odd number of incoming quanta carrying an odd fermion number. Since all the outgoing quanta are bosons, $(-1)^{F_L+F_R}$ cannot be conserved. In other words, this S-matrix is not unitary! This means that the theory should have more asymptotic states in addition to the quanta we mentioned here. For example, it is possible that a coherent state of C_+ quanta carries the necessary fermion number. An explicit example of this phenomenon which is very similar to our case is given in [22, 23]. Such behavior of the S-matrix, where nonunitarity in perturbative calculations is fixed in the full theory, is reminiscent of black hole physics, and is therefore worthy of further study. A possible interesting direction is to look for a matrix model of this system (for first attempts see [10, 11, 12]).

2.4 IIA

An orbifold of the IIB theory by $(-1)^{F_L}$ (or alternatively, an orbifold of the 0A theory by $(-1)^{f_L}$) leads to the IIA theory. More precisely, here the transformations $(-1)^{F_L+f_R} = (-1)^{f_L+f_R+F_R} = 1$ act trivially, and $(-1)^{F_L}$ and $(-1)^{F_R}$ generate the symmetry G. The spectrum includes

$$
\begin{aligned}
\Psi_-(p) &= e^{-\frac{\varphi}{2}-\overline{\varphi}-i\frac{H}{2}+ip(x+\overline{x})+(1-|p|)\phi} && p \le 0 \\
\Psi_+(p) &= e^{-\varphi-\frac{\overline{\varphi}}{2}+i\frac{\overline{H}}{2}+ip(x+\overline{x})+(1-|p|)\phi} && p \ge 0 \\
F^+ &= e^{-\frac{\varphi}{2}-\frac{\overline{\varphi}}{2}+\frac{i}{2}(H-\overline{H})+\phi}
\end{aligned} \tag{2.13}
$$

Here F^+ is the single RR flux of the theory. It is the field strength of a gauge field. The spectrum of particles is a single left moving and right moving Majorana fermion $\Psi_\pm$. Consider the parity transformations (1.2). This theory is invariant only under the

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$\mathbb{Z}_2$ generated by the combined operation $P_{ws}P_{ts}$. Using (1.3) (1.4), the two symmetries $(-1)^{F_{L,R}}$ which we mentioned above are related by conjugation by $P_{ws}P_{ts}$. Acting with either P_{ws} or P_{ts} leads to another isomorphic theory (IIA').

As in the IIB theory, the projection in the R-NS and the NS-R sectors is opposite to the ones in ten dimensions, where $(-1)^{f_L} = (-1)^{f_R+F_R} = 1$. That projection would not have allowed any operators from the R-NS and NS-R sectors in two dimensions and presumably would have resulted in an inconsistent theory.

Here, as in the IIB theory, there is only one translation invariant deformation of the theory — the RR flux q_+F^+.

Again, we have the operators (2.2) (2.3). The ground ring is a quotient of the IIB ground ring by $(-1)^{F_L}$ (or equivalently of the 0A ground ring by $(-1)^{f_L}$). It is freely generated by $O_+ = R_+^2$ and $O_- = R_-^2$. The fermions $\Psi_-(p)$ are in modules of this ring with the module relation $O_+ = 0$, while the fermions $\Psi_+(p)$ are in modules of this ring with the module relation $O_- = 0$. As in the IIB discussion, we expect a matrix model of this system to exist, where $\Psi_\pm(p)$ correspond to ripples on different halves of the Fermi surface.

As in the IIB theory, the S-matrix of this theory is interesting. The incoming quanta are the fermions Ψ_- and the outgoing quanta are the fermions Ψ_+. In the absence of background RR flux q_+F^+ the S-matrix of these quanta is not unitary without possible coherent states. Unlike the IIB theory, there is a possible unitary scattering when RR flux is turned on. Clearly, this S-matrix should be further explored, perhaps by finding a matrix model for this system.

3. The theories on $\mathbb{R} \times S^1$

These four theories can be compactified on a circle with various twists by their symmetry elements. We will limit ourselves to compactifications without twists by the two parity transformations (1.2), and will also not consider possible asymmetric orbifolds. A priori each theory can be twisted by any of the four elements of its $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. However, since in all our cases two of these elements are related by conjugation by a certain parity symmetry of the theory, only three of these compactifications are distinct. Finally, T-dualities relate these 12 compactified theories leaving only eight independent ones.

All the theories have the currents (2.2) and the dilaton/graviton operator (2.3), which changes the radius R. We will not list them explicitly except in special cases. There are also various discrete states and ring elements, most of them will not be discussed.

The torus partition function of these theories are divergent both in the weak coupling and the strong coupling regions. But its value per unit length is finite. This value is not easy to compute because of subtleties in the odd spin structures. In [4], where our first three examples were studied in detail, it was argued that the torus partition function can be obtained by summing over the various physical modes using ζ-function regularization; i.e. $\sum n \to -\frac{1}{12}$, $\sum 2n \to -\frac{2}{12}$ and $\sum 2n+1 \to +\frac{1}{12}$. In doing it we should remember to include target space fermions with an extra minus sign. We will use this procedure below.

The various compactifications are (in order not to clutter the equations, we suppressed the dependence on ϕ):

Line 1: 0A/0B on a circle. 0A compactified on a circle of radius R (without a twist) is T-dual to 0B on $2/R$. The spectrum is

$$
\begin{aligned}
T(p) &= e^{-\varphi-\overline{\varphi}+i\frac{p}{R}(x+\overline{x})} \qquad p=0,\pm1,\pm2,\dots \\
\widetilde{T}(w) &= e^{-\varphi-\overline{\varphi}+i\frac{wR}{2}(x-\overline{x})} \qquad w=\pm1,\pm2,\dots \\
\widetilde{C}(w) &= e^{-\frac{\varphi}{2}-\frac{\overline{\varphi}}{2}+\frac{i\epsilon(w)}{2}(H-\overline{H})+i\frac{wR}{2}(x-\overline{x})} \qquad w=\pm1,\pm2,\dots \\
F^{\pm} &= \widetilde{C}_{\pm} = e^{-\frac{\varphi}{2}-\frac{\overline{\varphi}}{2}\pm\frac{i}{2}(H-\overline{H})}
\end{aligned}
\tag{3.1}
$$

Here we denoted by tilde the fields of the noncompact T-dual theory.

The torus partition function is

$$
\mathcal{Z}_1 = -\frac{1}{12}\left(\frac{1}{R}+R\right) \tag{3.2}
$$

In addition to the three deformations of the 0A or 0B theories on $\mathbb{R}$, here we can preserve one of the shift symmetries around the circle by turning on operators with either nonzero momentum or nonzero winding. If such an operator is T or $\widetilde{T}$, it leads to a Sine-Liouville interaction on the worldsheet. We will not explore it here.

Line 2: super-affine 0B. This is a compactification of the 0B theory such that as we move around a circle of radius R we twist by $(-1)^{F_L}$; i.e. we mod out the theory on a circle with radius[3] $2R$ by $(-1)^{F_L}e^{i\pi p}$. The spectrum is

$$
\begin{aligned}
T(p) &= e^{-\varphi-\overline{\varphi}+i\frac{p}{R}(x+\overline{x})} \qquad p=0,\pm1,\pm2,\dots \\
\widetilde{T}(w) &= e^{-\varphi-\overline{\varphi}+i\frac{wR}{2}(x-\overline{x})} \qquad w=\pm2,\pm4,\dots \\
C(p) &= e^{-\frac{\varphi}{2}-\frac{\overline{\varphi}}{2}+\frac{i\epsilon(p)}{2}(H+\overline{H})+i\frac{p}{R}(x+\overline{x})} \qquad p=\pm\frac{1}{2},\pm\frac{3}{2},\dots \\
\widetilde{C}(w) &= e^{-\frac{\varphi}{2}-\frac{\overline{\varphi}}{2}+\frac{i\epsilon(w)}{2}(H-\overline{H})+i\frac{wR}{2}(x-\overline{x})} \qquad w=\pm1,\pm3,\dots
\end{aligned}
\tag{3.3}
$$

These theories are selfdual under the transformation $R\to1/R$. At the selfdual radius $R=1$ the operators (2.2) are accompanied by two more left moving and two more right moving worldsheet currents leading to $\mathrm{SU}(2)\times\mathrm{SU}(2)$ symmetry:

$$
\begin{aligned}
j^{\pm} &= e^{-\varphi\pm ix} \\
j^{0} &= e^{-\varphi}\psi_x \\
\overline{j}^{\pm} &= e^{-\overline{\varphi}\pm i\overline{x}} \\
\overline{j}^{0} &= e^{-\overline{\varphi}}\overline{\psi}_x\,.
\end{aligned}
\tag{3.4}
$$

All the physical operators are in representations of this symmetry. For example, the dilaton/graviton operator (2.3), the operators $T(p=\pm1)$, $\widetilde{T}(w=\pm2)$ in (3.3) and

[3] It is common in the literature to denote by R the radius of this circle; i.e. $R_{\text{there}}=2R_{\text{here}}$.

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four other operators form the $(\mathbf{1},\mathbf{1})$ representation. They can be written as $j^a\overline{j}^b$ with $a,b=\pm,0$. Similarly, $(C(p=\pm\frac{1}{2}),\widetilde{C}(w=\pm1))$ are in the $(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})$ representation. Away from the selfdual point the operators $j^\pm$ and $\overline{j}^\pm$ do not deform to physical operators and they simply disappear.

The torus partition function is

$$\mathcal{Z}_2=-\frac{1}{24}\left(\frac{1}{R}+R\right) \tag{3.5}$$

Finally, note that the twist around the circle eliminates the zero momentum deformations (2.7). This means that this compactification is possible only when $q_+=q_-=0$.

Line 3: super-affine 0A. This is a compactification of the 0A theory such that as we move around a circle of radius R we twist by $(-1)^{F_L}$; i.e. we mod out the theory on a circle with radius $2R$ by $(-1)^{F_L}e^{i\pi p}$. The spectrum is

$$\begin{aligned} T(p) &= e^{-\varphi-\overline{\varphi}+i\frac{p}{R}(x+\overline{x})} && p=0,\pm1,\pm2,\dots \\ \widetilde{T}(w) &= e^{-\varphi-\overline{\varphi}+i\frac{wR}{2}(x-\overline{x})} && w=\pm2,\pm4,\dots \end{aligned} \tag{3.6}$$

These theories are selfdual under the transformation $R\to1/R$, and at the selfdual radius $R=1$ have the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ currents $j^{\pm,0}$, $\overline{j}^{\pm,0}$ of (3.4) and the vertex operators $j^a\overline{j}^b$. The torus partition function is

$$\mathcal{Z}_3=-\frac{1}{12}\left(\frac{1}{R}+R\right) \tag{3.7}$$

and as in the 0B super-affine line, this compactification is possible only when $q_+=q_-=0$.

Line 4: IIA/IIB on a circle. IIA compactified on a circle with radius R (without a twist) is T-dual to IIB on $\frac{2}{R}$. The physical operators are

$$\begin{aligned} \Psi_-(p) &= e^{-\frac{\varphi}{2}-\overline{\varphi}-i\frac{H}{2}+i\frac{p}{R}(x+\overline{x})} && p=0,-1,-2,\dots \\ \widetilde{\Psi}_-(w) &= e^{-\frac{\varphi}{2}-\overline{\varphi}-i\frac{H}{2}+i\frac{wR}{2}(x-\overline{x})} && w=-1,-2,\dots \\ \Psi_+(p) &= e^{-\varphi-\frac{\overline{\varphi}}{2}+i\frac{\overline{H}}{2}+i\frac{p}{R}(x+\overline{x})} && p=0,1,2,\dots \\ \widetilde{\overline{\Psi}}_-(w) &= e^{-\varphi-\frac{\overline{\varphi}}{2}+i\frac{\overline{H}}{2}+i\frac{wR}{2}(x-\overline{x})} && w=-1,-2,\dots \\ \widetilde{C}_+(w) &= e^{-\frac{\varphi}{2}-\frac{\overline{\varphi}}{2}+\frac{i}{2}(H-\overline{H})+i\frac{wR}{2}(x-\overline{x})} && w=0,1,2,\dots \end{aligned} \tag{3.8}$$

The operator $\widetilde{C}(0)=F$ is the RR flux. It can be added to the worldsheet lagrangian without breaking the two translation symmetries of the compactification. The torus partition function is

$$\mathcal{Z}_4=+\frac{1}{24}\left(\frac{2}{R}+\frac{R}{2}\right). \tag{3.9}$$

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Line 5: super-affine IIB. Here the IIB theory is compactified on a circle of radius R twisted by $(-1)^{F_L}$. It is selfdual with $R \to 1/R$. The spectrum is

$$\begin{aligned}
\Psi_-(p) &= e^{-\frac{\varphi}{2}-\overline{\varphi}-i\frac{H}{2}+i\frac{p}{R}(x+\overline{x})} && p = -\frac{1}{2}, -\frac{3}{2}, \dots \\
\widetilde{\Psi}_+(w) &= e^{-\frac{\varphi}{2}-\overline{\varphi}+i\frac{H}{2}+i\frac{wR}{2}(x-\overline{x})} && w = 1, 3, \dots \\
\overline{\Psi}_-(p) &= e^{-\varphi-\frac{\overline{\varphi}}{2}-i\frac{\overline{H}}{2}+i\frac{p}{R}(x+\overline{x})} && p = 0, -1, -2, \dots \\
\widetilde{\overline{\Psi}}_+(w) &= e^{-\varphi-\frac{\overline{\varphi}}{2}-i\frac{\overline{H}}{2}+i\frac{wR}{2}(x-\overline{x})} && w = 2, 4, \dots \\
C_+(p) &= e^{-\frac{\varphi}{2}-\frac{\overline{\varphi}}{2}+\frac{i}{2}(H+\overline{H})+i\frac{p}{R}(x+\overline{x})} && p = \frac{1}{2}, \frac{3}{2}, \dots \\
\widetilde{C}_-(w) &= e^{-\frac{\varphi}{2}-\frac{\overline{\varphi}}{2}-\frac{i}{2}(H-\overline{H})+i\frac{wR}{2}(x-\overline{x})} && w = -1, -3, \dots
\end{aligned} \tag{3.10}$$

As in the super-affine lines of the type 0 theories, at the selfdual radius $R = 1$ we have enhanced symmetry. But this time the symmetry is $\mathrm{SU}(2) \times \mathrm{U}(1)$ with the currents $j^{\pm,0}$ and $\overline{j}^0$ (without $\overline{j}^{\pm}$) of (3.4). The spectrum is in representations of this symmetry. For example, $(\Psi_-(p = -\frac{1}{2}), \widetilde{\Psi}_+(w = 1))$ and $(C_+(p = \frac{1}{2}), \widetilde{C}_-(w = -1))$ are doublet with U(1) charges $-\frac{1}{2}$ and $+\frac{1}{2}$ respectively.. Using the currents we also find the three NS-NS vertex operators $j^a\overline{j}^0$.

The torus partition function is

$$\mathcal{Z}_5 = +\frac{1}{24}\left(\frac{1}{R} + R\right) \tag{3.11}$$

Line 6: super-affine IIA. Here the IIA theory is compactified on a circle of radius R twisted by $(-1)^{F_L}$. It is selfdual with $R \to 1/R$. The spectrum is

$$\begin{aligned}
\Psi_-(p) &= e^{-\frac{\varphi}{2}-\overline{\varphi}-i\frac{H}{2}+i\frac{p}{R}(x+\overline{x})} && p = -\frac{1}{2}, -\frac{3}{2}, \dots \\
\widetilde{\Psi}_+(w) &= e^{-\frac{\varphi}{2}-\overline{\varphi}+i\frac{H}{2}+i\frac{wR}{2}(x-\overline{x})} && w = 1, 3, \dots \\
\Psi_+(p) &= e^{-\varphi-\frac{\overline{\varphi}}{2}+i\frac{\overline{H}}{2}+i\frac{p}{R}(x+\overline{x})} && p = 0, 1, 2, \dots \\
\widetilde{\overline{\Psi}}_-(w) &= e^{-\varphi-\frac{\overline{\varphi}}{2}+i\frac{\overline{H}}{2}+i\frac{wR}{2}(x-\overline{x})} && w = -2, -4, \dots
\end{aligned} \tag{3.12}$$

As in the super-affine IIB line, at the selfdual radius $R = 1$ we have the symmetry $\mathrm{SU}(2) \times \mathrm{U}(1)$ with the currents $j^{\pm,0}$ and $\overline{j}^0$ (without $\overline{j}^{\pm}$) of (3.4) and the three vertex operators $j^a\overline{j}^0$.

The torus partition function is

$$\mathcal{Z}_6 = +\frac{1}{48}\left(\frac{1}{R} + R\right). \tag{3.13}$$

Line 7: IIB on a thermal circle. Here we compactify the IIB theory on a thermal circle of radius R; i.e. we twist with $(-1)^{F_L+F_R}$ (or equivalently, with $(-1)^{f_L}$). It is T-dual to 0A twisted by $(-1)^{f_L}$ on $\frac{2}{R}$. The physical operators are

$$\widetilde{T}(w) = e^{-\varphi-\overline{\varphi}+i\frac{wR}{2}(x-\overline{x})} \qquad w = \pm 1, \pm 3, \dots$$

$$
\begin{aligned}
\Psi_-(p) &= e^{-\frac{\varphi}{2}-\overline{\varphi}-i\frac{H}{2}+i\frac{p}{R}(x+\overline{x})} \qquad p=-\frac{1}{2},-\frac{3}{2},\dots \\
\overline{\Psi}_-(p) &= e^{-\varphi-\frac{\overline{\varphi}}{2}-i\frac{\overline{H}}{2}+i\frac{p}{R}(x+\overline{x})} \qquad p=-\frac{1}{2},-\frac{3}{2},\dots \\
C_+(p) &= e^{-\frac{\varphi}{2}-\frac{\overline{\varphi}}{2}+\frac{i}{2}(H+\overline{H})+i\frac{p}{R}(x+\overline{x})} \qquad p=0,1,2,\dots
\end{aligned} \tag{3.14}
$$

For $R=1$ we have a phenomenon similar to (3.4) but in the Ramond sector. Here we find the worldsheet currents $\mathcal{S}_+ = e^{-\frac{\varphi}{2}+i\frac{H}{2}+ix}$ and $\overline{\mathcal{S}}_+ = e^{-\frac{\overline{\varphi}}{2}+i\frac{\overline{H}}{2}+i\overline{x}}$, which lead to fermionic target space symmetries with the algebra $\{\mathcal{S}_+,\mathcal{S}_+\} = \{\overline{\mathcal{S}}_+,\overline{\mathcal{S}}_+\} = 0$. Using these operators we find the "gravitino" vertex operators $\mathcal{S}_+ e^{-\overline{\varphi}}\overline{\psi}_x$ and $\overline{\mathcal{S}}_+ e^{-\varphi}\psi_x$. Note that $C_+(p=1) = \mathcal{S}_+\overline{\mathcal{S}}_+$. This theory coincides with the construction of [6]. As with (3.4), away from $R=1$ the operators $\mathcal{S}_+$ and $\overline{\mathcal{S}}_+$ do not deform to physical operators.

The torus partition function is

$$
\mathcal{Z}_7 = -\frac{1}{12}\left(\frac{1}{R}-\frac{R}{2}\right). \tag{3.15}
$$

Line 8: IIA on a thermal circle. Here we compactify the IIA theory on a thermal circle of radius R; i.e. we twist with $(-1)^{F_L+F_R}$ (or equivalently, with $(-1)^{f_L}$). It is T-dual to 0B twisted by $(-1)^{f_l}$ on $\frac{2}{R}$. The physical operators are

$$
\begin{aligned}
\widetilde{T}(w) &= e^{-\varphi-\overline{\varphi}+i\frac{wR}{2}(x-\overline{x})} \qquad w=\pm1,\pm3,\dots \\
\Psi_-(p) &= e^{-\frac{\varphi}{2}-\overline{\varphi}-i\frac{H}{2}+i\frac{p}{R}(x+\overline{x})} \qquad p=-\frac{1}{2},-\frac{3}{2},\dots \\
\Psi_+(p) &= e^{-\varphi-\frac{\overline{\varphi}}{2}+i\frac{\overline{H}}{2}+i\frac{p}{R}(x+\overline{x})} \qquad p=\frac{1}{2},\frac{3}{2},\dots \\
\widetilde{C}_-(w) &= e^{-\frac{\varphi}{2}-\frac{\overline{\varphi}}{2}-\frac{i}{2}(H-\overline{H})+i\frac{wR}{2}(x-\overline{x})} \qquad w=-1,-3,\dots \\
\widetilde{C}_+(w) &= e^{-\frac{\varphi}{2}-\frac{\overline{\varphi}}{2}+\frac{i}{2}(H-\overline{H})+i\frac{wR}{2}(x-\overline{x})} \qquad w=0,2,\dots
\end{aligned} \tag{3.16}
$$

For $R=1$ there are worldsheet currents $\mathcal{S}_+ = e^{-\frac{\varphi}{2}+i\frac{H}{2}+ix}$ and $\overline{\mathcal{S}}_- = e^{-\frac{\overline{\varphi}}{2}-i\frac{\overline{H}}{2}-i\overline{x}}$ which lead to fermionic target space symmetries with the algebra $\{\mathcal{S}_+,\mathcal{S}_+\} = \{\overline{\mathcal{S}}_-,\overline{\mathcal{S}}_-\} = 0$. Using these operators we find the "gravitino" vertex operators $\mathcal{S}_+ e^{-\overline{\varphi}}\overline{\psi}_x$ and $\overline{\mathcal{S}}_- e^{-\varphi}\psi_x$. Note that $C_+(w=2) = \mathcal{S}_+\overline{\mathcal{S}}_-$. This theory coincides with the construction of [6]. As with (3.4), away from $R=1$ the operators $\mathcal{S}_+$ and $\overline{\mathcal{S}}_-$ disappear.

The torus partition function is

$$
\mathcal{Z}_8 = -\frac{1}{24}\left(\frac{1}{R}-\frac{R}{2}\right). \tag{3.17}
$$

4. The branes

Here we consider the analog of the FZZT branes [24, 25] of our four theories on $\mathbb{R}^2$. We will use the fermionic string version of the FZZT branes, which were discussed in [26]–[30].

Since we have limited ourselves to the linear dilaton theory without a "Liouville wall", the results are going to be meaningful only when the branes dissolve at the weak coupling region. For the type 0 theories we can find the branes by considering the branes of the type 0 theory for nonzero μ in the limit $\mu \to 0$ with fixed μ_B. These branes are expressed in terms of the parameter σ. From the worldsheet point of view this is the Dirichlet boundary condition on the Backlund field [29], and from the matrix model point of view, where μ_B is the eigenvalue coordinate, σ is essentially the "time of flight variable." For the different branes $\mu_B \sim \sqrt{|\mu|}\cosh\sigma$ or $\mu_B \sim \sqrt{|\mu|}\sinh\sigma$. Therefore, we are interested in the limit $\mu \to 0$, $|\sigma| \to \infty$ with $\mu_B \sim \sqrt{|\mu|}e^{|\sigma|}$ fixed. In this limit the expressions for the boundary states simplify.

Consider first the 0B theory. Here we find 6 branes. Two of them are RR neutral and satisfy Neumann boundary conditions in euclidean time ($X = x + \overline{x}$). They differ by the boundary conditions of the supercharge $Q = \eta\overline{Q}$ with $\eta = \pm 1$. There are also four RR charged branes labelled by $\eta = \pm 1$ and $\xi = \pm 1$ (ξ distinguishes branes and anti-branes), which have Dirichlet boundary conditions in euclidean time. Taking $\mu \to 0$, the boundary states are

$$\begin{aligned} |\mu_B, 0, \eta\rangle &= \sqrt{2}\int_0^\infty dp\ \mu_B^{-2ip} A_{NS}(p)|NS, p, \partial X = 0, \eta\rangle \\ |\mu_B, X_0, \xi, \eta\rangle &= \int_0^\infty dp\ \mu_B^{-2ip}\left(A_{NS}(p)|NS, p, X = X_0, \eta\rangle + \xi A_R(p)|R, p, X = X_0, \eta\rangle\right) \end{aligned} \tag{4.1}$$

where p is the Liouville momentum, $A_{NS}(p)$ and $A_R(p)$ are momentum dependent functions, whose form will not be important in this discussion, $|NS, p, \partial X = 0, \eta\rangle$ and $|NS, p, X = X_0, \eta\rangle$ are NS Ishibashi state with Liouville momentum p and Neumann or Dirichlet boundary conditions for X. They include the boundary states for the ghosts. Similarly, $|R, p, X = X_0, \eta\rangle$ are R Ishibashi states. The relative factor of $\sqrt{2}$ between the neutral and charged branes is common. It is familiar from the study of Cardy states in the Ising model. It guarantees that the lowest open string tachyon appears once on the brane. Note that the only dependence on μ_B is through the simple factor μ_B^{-2ip} which reproduces KPZ scaling (recall that $\mu = 0$ and hence the only dimensional parameter is μ_B).

Clearly, the symmetry $(-1)^{F_L}$ leaves the neutral brane invariant and exchanges the charged brane with its anti-brane, $\xi \to -\xi$. The symmetry $(-1)^{f_L}$ changes the sign of the boundary conditions of the supercharge and maps $\eta \to -\eta$.

The actual branes are obtained by exponentiating the boundary states (4.1). Since in the asymptotic region the theory has an $SU(2)\times SU(2)$ symmetry associated with the RR scalar $C = C_+ + C_-$, the branes are in representations of this symmetry. As always, the charge of the brane is localized near the point it dissolves ($X = X_0, \phi \sim \log|\mu_B|$). Clearly, the neutral branes are in singlets, and the four charged branes are in $(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}})$.[4] These latter branes can be thought of as insertions of target space vertex operators

$$e^{\frac{i\xi}{\sqrt{2}}(\eta C_+ + C_-)} \tag{4.2}$$

times a function of T at the point the brane dissolves.

[4]The fact that they carry charges $\pm\frac{1}{2}$ is consistent with the discussion in [30] about the charges of such branes.

In the 0A theory the neutral and charged branes are reversed. There are two neutral branes which are localized in X and four charged branes which have Neumann boundary conditions in X. The latter are charged under the two gauge fields of the type 0A theory, and their charges are localized near the point where the branes dissolve.

The two type-II theories are obtained by moding out the type 0 theories by $(-1)^{f_L}$. Therefore, the branes in the type-IIB theory are

$$\begin{aligned} |\mu_B, 0\rangle &= \frac{1}{\sqrt{2}} \left(|\mu_B, 0, \eta = +1\rangle - |\mu_B, 0, \eta = -1\rangle \right) \\ |\mu_B, X_0, \xi\rangle &= \frac{1}{\sqrt{2}} \left(|\mu_B, X_0, \xi, \eta = +1\rangle - |\mu_B, X_0, \xi, \eta = -1\rangle \right). \end{aligned} \tag{4.3}$$

Here the relative sign of the two branes of the type 0B theory is such that the closed string NS-NS tachyon is projected out. Therefore, the only physical NS-NS states which couple to these branes are discrete states like the dilaton/graviton operator $V = e^{-\varphi - \overline{\varphi}} \psi_x \overline{\psi}_x$. Similarly, only one chirality of the RR field, C_+ couples to the brane. Finally, the normalization factor $\frac{1}{\sqrt{2}}$ is such that there is only one open string tachyon on the brane. Because of this normalization, the analog of (4.2) is

$$e^{i\xi C_+} \tag{4.4}$$

i.e. it is like a free fermion. This suggests that the projection from 0B to IIB does not simply remove C_-, but it also changes the radius of C_+.

The branes of the type-IIA theory are similar. There is a single neutral brane which is localized in X and a charged brane and its anti-brane which are stretched in X. The charged branes are charged under the single gauge field of the IIA theory.

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